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A one-dimensional model for ionization induced by scattering with a heavy particle

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Abstract

The ionization probability for a one-dimensional model atom perturbed by a moving repulsive scatterer is considered. The moving scattering centre is meant to mimic a second quantum particle crossing the region where a much lighter particle is initially bound in the atom. We compute the first three terms (of order $t^{3/2}$, t^2, $t^{5/2}$ respectively) in the expansion for small times of the ionization probability and we deduce that the first term showing explicit dependence on the velocity of the scatterer is of order $t^{5/2}$. A possible application of the model for the description of a quantum particle in a cloud chamber is also outlined.

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1. Introduction

In this paper, we consider the problem of the ionization of a quantum particle of mass m, which at time zero is in a bound state, due to the interaction with another quantum particle with a large mass M.

We shall restrict ourselves to a one-dimensional model and we assume that the binding potential for the particle of mass m is given by an attractive δ-interaction placed at the origin.

The interaction with the heavy particle is phenomenologically described through a time-dependent δ-interaction centred at a point which is moving with constant velocity. The model is meant to be an approximation of the complete two-body problem when the mass M of the heavy particle is very large and its initial state is a properly chosen coherent state.

Under these conditions it is reasonable to expect that the heavy particle is well described by the free evolution of a wave packet while the initially bound particle only 'sees' the centre of the wave packet.

We plan to address the problem of justifying this approximation quantitatively in further work. For a general computational scheme in a different approach to an analogous scattering problem we also refer to [GS].

Here we shall limit ourselves to the analysis of the following evolution problem,

$$\mathrm{i}\frac{\partial \psi(t)}{\partial t} = H_\alpha \psi(t) + \beta\delta_{x(t)}\psi(t) \tag{1.1}$$

$$\psi(0) = \psi_\alpha \tag{1.2}$$

where

$$H_\alpha = -\tfrac{1}{2}\Delta + \alpha\delta_0 \tag{1.3}$$

$$\psi_\alpha(r) = \sqrt{|\alpha|}\, \mathrm{e}^{\alpha|r|}. \tag{1.4}$$

We denote by δ_y the Dirac measure placed at the point $y \in \mathbb{R}$ and we have chosen units in which $\hbar = m = 1$; moreover, we fix $\alpha < 0$, $\beta > 0$, $x(t) = x_0 + v_0 t$, with $x_0 < 0$, $v_0 \in \mathbb{R}$, and consider the evolution problem for $0 < t < \frac{|x_0|}{v_0}$.

Note that the initial state (1.4) is exactly the (unique) bound state of the self-adjoint Hamiltonian (1.3) corresponding to the eigenvalue $E = -\frac{\alpha^2}{2}$ (see, e.g., [AGH-KH]).

Problem (1.1), (1.2) is a linear non-autonomous Schrödinger evolution problem with the time-dependent generator $H(t) = H_\alpha + \beta\delta_{x(t)}$.

For each $t \geqslant 0$, such generator can be defined as the self-adjoint operator in $L^2(\mathbb{R})$ associated with the closed and bounded from below quadratic form

$$F(t)(u) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x |\nabla u|^2 + \alpha|u(0)|^2 + \beta|u(x(t))|^2, \qquad D(F(t)) = H^1(\mathbb{R}). \tag{1.5}$$

We remark that the form domain does not depend on time, while the operator domain does. On the other hand, the initial state (1.4) belongs to the form domain but it does not belong to the operator domain at time zero $D(H(0))$ (essentially, the reason is that (1.4) does not satisfy the boundary condition at x_0 corresponding to the point interaction $\beta\delta_{x_0}$).

As a consequence, we cannot expect to find a strong solution of problem (1.1), (1.2). In fact, we will prove the existence of a weak solution $\psi(t)$, in the sense of quadratic forms. By this we mean that $\psi(t) \in H^1(\mathbb{R})$ for $t > 0$ and

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(v, \psi(t)) = \mathcal{B}(v, \psi(t)) \tag{1.6}$$

for any $v \in H^1(\mathbb{R})$, where $\mathcal{B}(\cdot, \cdot)$ is the bilinear form associated with the quadratic form (1.5). We refer to [Si] for general sufficient conditions which guarantee existence and uniqueness of the solution in the sense of quadratic forms for non-autonomous evolution problems of this kind.

We observe that existence and uniqueness of the solution can also be derived following the line of the proof in [DFT], where the more delicate case of n moving point interactions with arbitrary velocity in dimension 3 is studied.

In the case of our specific evolution problem we are interested in a particular representation of the solution allowing an explicit computation of the asymptotic expansion for small times of the solution itself.

To this aim, we exploit the fact that the Hamiltonian H_α is solvable and in particular the corresponding unitary group $U_\alpha(t) \equiv \mathrm{e}^{-\mathrm{i}tH_\alpha}$ is explicitly known [S, ABD]. In fact, its integral kernel for $\alpha < 0$ is

$$U_\alpha(t; x, y) = U_0(t; x - y) + \mathrm{e}^{-\mathrm{i}Et}\psi_\alpha(x)\psi_\alpha(y) - |\alpha|\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-|\alpha|u} U_0(t; u - |x| - |y|) \tag{1.7}$$

where $x, y \in \mathbb{R}$ and $U_0(t;x)$ denotes the free unitary group

$$U_0(t;x) = \frac{\mathrm{e}^{-\frac{|x|^2}{2\mathrm{i}t}}}{\sqrt{2\pi\mathrm{i}t}}. \tag{1.8}$$

Then it is convenient to write the solution in the form

$$\psi(r,t) = (U_\alpha(t)\psi_\alpha)(r) - \mathrm{i}\beta\int_0^t \mathrm{d}s\, U_\alpha(t-s;r,x(s))\psi(x(s),s). \tag{1.9}$$

Using the fact that ψ_α is an eigenvector of H_α with eigenvalue E and defining $q(t) = \psi(x(t),t)$ we obtain the representation

$$\psi(r,t) = \mathrm{e}^{-\mathrm{i}Et}\psi_\alpha(r) - \mathrm{i}\beta\int_0^t \mathrm{d}s\, U_\alpha(t-s;r,x(s))q(s) \tag{1.10}$$

$$q(t) = \mathrm{e}^{-\mathrm{i}Et}\psi_\alpha(x(t)) - \mathrm{i}\beta\int_0^t \mathrm{d}s\, U_\alpha(t-s;x(t),x(s))q(s). \tag{1.11}$$

The integral equation (1.11) for $q(t)$ can be rewritten in a more explicit form exploiting (1.7) and considering $x(t) = x_0 + v_0 t$, with $0 < t < \frac{|x_0|}{v_0}$. One obtains

$$\begin{aligned} q(t) = q_0\,\mathrm{e}^{-\mathrm{i}Et+|\alpha|v_0 t} - \mathrm{i}\beta\int_0^t \mathrm{d}s\, q(s)\frac{\mathrm{e}^{\mathrm{i}\frac{v_0^2}{2}(t-s)}}{\sqrt{2\pi\mathrm{i}(t-s)}} - \mathrm{i}\beta q_0^2\,\mathrm{e}^{-\mathrm{i}Et+|\alpha|v_0 t}\int_0^t \mathrm{d}s\, q(s)\,\mathrm{e}^{\mathrm{i}Es+|\alpha|v_0 s} \\ + \mathrm{i}\beta|\alpha|\int_0^t \mathrm{d}s\frac{q(s)}{\sqrt{2\pi\mathrm{i}(t-s)}}\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-|\alpha|u}\exp\left(\mathrm{i}\frac{(u-2|x_0|+v_0(t+s))^2}{2(t-s)}\right) \end{aligned} \tag{1.12}$$

where

$$q_0 = \sqrt{|\alpha|}\,\mathrm{e}^{-|\alpha||x_0|}. \tag{1.13}$$

It is not hard to see that formulae (1.10), (1.12) define the unique solution of our evolution problem in the sense of quadratic forms.

Avoiding technical details, we only observe that the integral equation (1.12) is uniquely solvable in the space of continuous functions $C^0([0,T])$, for any fixed $T > 0$. Exploiting the smoothing properties of the integral operators in (1.12) one also obtains that $q \in C^1((0,T])$ and $\sup_{t\in[0,T]} t^{1/2}|\dot{q}(t)| < \infty$.

Then an explicit computation shows that for such $q(t)$ the rhs of (1.10) belongs to $H^1(\mathbb{R})$ for any $t \in [0,T]$ and (1.6) is satisfied.

In the next sections, we will study the small time behaviour of the ionization probability for the particle which is initially in the bound state ψ_α. Such probability is defined by the expression

$$\mathcal{P}(t) = 1 - |(\psi_\alpha, \psi(t))|^2. \tag{1.14}$$

Since the scalar product in (1.14) is explicitly given by (see (1.10))

$$(\psi_\alpha, \psi(t)) = \mathrm{e}^{-\mathrm{i}Et}(1 - A(t)) \tag{1.15}$$

$$A(t) = \mathrm{i}\beta q_0\int_0^t \mathrm{d}s\, q(s)\,\mathrm{e}^{\mathrm{i}Es+|\alpha|v_0 s} \tag{1.16}$$

we can write

$$\mathcal{P}(t) = 2\,\mathrm{Re}\,A(t) - |A(t)|^2. \tag{1.17}$$

In section 2 the asymptotic expansion of the solution $q(t)$ of (1.12) will be computed up to the order $t^{3/2}$.

Such a result will be used in section 3 to compute the small time asymptotics for $\mathcal{P}(t)$ up to the order $t^{5/2}$. We shall find that the coefficient of this last term is the first non-trivial coefficient of the expansion which is explicitly dependent on the velocity v_0.

2. Asymptotic expansion for $q(t)$

For the solution of the integral equation (1.12) a complete asymptotic expansion for small times can be given.

Equation (1.12) is a Volterra integral equation with a weakly singular kernel and the singularity is of the type $(t-s)^{-1/2}$ (see, e.g., [BH]). This implies that the solution $q(t)$ has an expansion of the form

$$q(t) \sim \sum_{k=0}^{\infty} q_k t^{k/2} \tag{2.1}$$

where all the coefficients $q_k \in \mathbb{C}$ can be explicitly computed. For reasons that will be clear in the following, we shall limit ourselves to the computation up to $k = 3$.

Proposition 2.1. *Let $q(t)$ be the (continuous) solution of (1.12); then*

$$q(t) = q_0 + q_1 t^{1/2} + q_2 t + q_3 t^{3/2} + O(t^2) \tag{2.2}$$

where q_0 is given by (1.13) and

$$q_1 = -\frac{2\mathrm{i}\beta}{\sqrt{2\pi \mathrm{i}}} q_0 \tag{2.3}$$

$$q_2 = \left[|\alpha| v_0 + \mathrm{i}\left(\frac{\beta^2}{2} - E \right) \right] q_0 \tag{2.4}$$

$$q_3 = \frac{\beta v_0^2}{3\sqrt{2\pi \mathrm{i}}} q_0. \tag{2.5}$$

Proof. Equation (1.12) for $q(t)$ can be rewritten in the form

$$q(t) = f(t) + \sum_{j=1}^{3} (K_j q)(t) \tag{2.6}$$

where

$$f(t) = q_0\, \mathrm{e}^{-\mathrm{i}Et + |\alpha| v_0 t} \tag{2.7}$$

$$(K_j q)(t) = \int_0^1 \mathrm{d}\sigma\, q(t(1-\sigma)) k_j(t,\sigma) \qquad j = 1, 2, 3 \tag{2.8}$$

$$k_1(t,\sigma) = c_1 t^{1/2} \frac{\mathrm{e}^{\mathrm{i}\frac{v_0^2}{2} t\sigma}}{\sigma^{1/2}} \qquad c_1 = -\frac{\mathrm{i}\beta}{\sqrt{2\pi \mathrm{i}}} \tag{2.9}$$

$$k_2(t,\sigma) = c_2 t\, \mathrm{e}^{-\mathrm{i}Et\sigma + |\alpha| v_0 t(2-\sigma)} \qquad c_2 = -\mathrm{i}\beta q_0^2 \tag{2.10}$$

$$k_3(t,\sigma) = c_3 \frac{t^{1/2}}{\sigma^{1/2}} \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-|\alpha| u} \exp\left(\mathrm{i} \frac{(u - 2|x_0| + v_0 t(2-\sigma))^2}{2t\sigma} \right) \qquad c_3 = \frac{\mathrm{i}\beta|\alpha|}{\sqrt{2\pi \mathrm{i}}}. \tag{2.11}$$

First we observe that one obviously has

$$\lim_{t\to 0} q(t) = \lim_{t \to 0} f(t) = q_0. \tag{2.12}$$

For the computation of the other coefficients of the expansion it is convenient to rewrite K_1q as

$$(K_1q)(t) = c_1 t^{1/2}\left[2q_0 + \int_0^1 \mathrm{d}\sigma \frac{q(t(1-\sigma)) - q_0}{\sqrt{\sigma}} + q_0 \int_0^1 \mathrm{d}\sigma \frac{\mathrm{e}^{\mathrm{i}\frac{v_0^2}{2}t\sigma} - 1}{\sqrt{\sigma}}\right.$$
$$\left. + \int_0^1 \mathrm{d}\sigma (q(t(1-\sigma)) - q_0)\frac{\mathrm{e}^{\mathrm{i}\frac{v_0^2}{2}t\sigma} - 1}{\sqrt{\sigma}}\right]. \tag{2.13}$$

For K_2q we have

$$(K_2q)(t) = -\mathrm{i}\beta q_0^2 t \int_0^1 \mathrm{d}\sigma\, q(t(1-\sigma)) + h_2(t) \tag{2.14}$$

where $h_2(t) = O(t^2)$.

Concerning K_3q we can write

$$k_3(t,\sigma) = c_3 \exp\left(\mathrm{i}\left[v_0^2 t\frac{(2-\sigma)^2}{2\sigma} - 2\frac{2-\sigma}{\sigma}|x_0|v_0\right]\right)\frac{t^{1/2}}{\sigma^{1/2}}$$
$$\times \int_0^\infty \mathrm{d}u \exp\left(-\left(|\alpha| - \mathrm{i}v_0\frac{2-\sigma}{\sigma}\right)u\right)\exp\left(\mathrm{i}\frac{(u-2|x_0|)^2}{2t\sigma}\right). \tag{2.15}$$

A complete asymptotic expansion of the integral in (2.15) for $t \to 0$ can be given (see, e.g., [ABD]). For the convenience of the reader, we give here the explicit computation of the first two terms of the expansion, which are relevant for our proof.

Introducing the change of variable $z = u - 2|x_0|$ we have

$$k_3(t,\sigma) = c_3 \exp\left(\mathrm{i}v_0^2 t\frac{(2-\sigma)^2}{2\sigma} - 2|\alpha||x_0|\right)\frac{t^{1/2}}{\sigma^{1/2}}$$
$$\times \int_{-2|x_0|}^\infty \mathrm{d}z \exp\left(-\left(|\alpha| - \mathrm{i}v_0\frac{2-\sigma}{\sigma}\right)z + \mathrm{i}\frac{z^2}{2t\sigma}\right). \tag{2.16}$$

Exploiting a standard stationary phase argument (i.e., repeated integration by parts) one obtains

$$\int_{-2|x_0|}^\infty \mathrm{d}z \exp\left(-\left(|\alpha| - \mathrm{i}v_0\frac{2-\sigma}{\sigma}\right)z + \mathrm{i}\frac{z^2}{2t\sigma}\right)$$
$$= \sqrt{2\pi\mathrm{i}}\sigma^{1/2}t^{1/2} - \mathrm{i}\frac{\exp\left(-\left(|\alpha| - \mathrm{i}v_0\frac{2-\sigma}{\sigma}\right) + \mathrm{i}\frac{2x_0^2}{t\sigma}\right)}{2|x_0|}\sigma t + g(t,\sigma) \tag{2.17}$$

where g is bounded and $g(t,\sigma) = O(t^{3/2})$ for almost all σ.

Substituting (2.17) in (2.16) we have

$$(K_3q)(t) = \mathrm{i}\beta q_0^2 t \int_0^1 \mathrm{d}\sigma\, q(t(1-\sigma))$$
$$- \mathrm{i}\frac{c_3}{2|x_0|}\,\mathrm{e}^{-4|\alpha||x_0|}t^{3/2}\int_0^1 \mathrm{d}\sigma\, q(t(1-\sigma))\sigma^{1/2}\,\mathrm{e}^{\mathrm{i}v_0^2 t\frac{(2-\sigma)^2}{2\sigma}}\,\mathrm{e}^{\mathrm{i}2|x_0|v_0\frac{(2-\sigma)}{\sigma}}\,\mathrm{e}^{\mathrm{i}\frac{2x_0^2}{t\sigma}} + h_3(t)$$
$$= \mathrm{i}\beta q_0^2 t\int_0^1 \mathrm{d}\sigma\, q(t(1-\sigma)) - \mathrm{i}\frac{c_3}{2|x_0|}\,\mathrm{e}^{-4|\alpha||x_0|}t^{3/2}q_0\int_0^1 \mathrm{d}\sigma\,\sigma^{1/2}\,\mathrm{e}^{\mathrm{i}2|x_0|v_0\frac{(2-\sigma)}{\sigma}}\,\mathrm{e}^{\mathrm{i}\frac{2x_0^2}{t\sigma}}$$
$$- \mathrm{i}\frac{c_3}{2|x_0|}\,\mathrm{e}^{-4|\alpha||x_0|}t^{3/2}\int_0^1 \mathrm{d}\sigma\left(q(t(1-\sigma))\,\mathrm{e}^{\mathrm{i}v_0^2 t\frac{(2-\sigma)^2}{2\sigma}} - q_0\right)$$
$$\times\,\sigma^{1/2}\,\mathrm{e}^{\mathrm{i}2|x_0|v_0\frac{(2-\sigma)}{\sigma}}\,\mathrm{e}^{\mathrm{i}\frac{2x_0^2}{t\sigma}} + h_3(t) \tag{2.18}$$

where $h_3(t) = O(t^2)$.

It is remarkable that the first terms in the rhs of (2.14) and (2.18) cancel exactly. This phenomenon is typical of the unitary propagator for a Hamiltonian with an attractive point interaction (for further comments on this kind of cancellation we refer the reader to [ABD]).

Moreover, we observe that the second term in the rhs of (2.18) is of order $t^{5/2}$, due to the presence of the rapidly oscillating phase under the integral sign. In fact,

$$\int_0^1 \mathrm{d}\sigma\, \sigma^{1/2}\, \mathrm{e}^{\mathrm{i}2|x_0|v_0 \frac{(2-\sigma)}{\sigma}}\, \mathrm{e}^{\mathrm{i}\frac{2x_0^2}{t\sigma}} = \int_1^\infty \mathrm{d}\nu \frac{1}{\nu^{5/2}}\, \mathrm{e}^{2\mathrm{i}x_0 v_0(2\nu-1)}\, \mathrm{e}^{2\mathrm{i}\frac{x_0^2}{t}\nu}$$
$$= \frac{\mathrm{i}t}{2x_0^2}\int_1^\infty \mathrm{d}\nu \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu^{5/2}}\, \mathrm{e}^{2\mathrm{i}x_0 v_0(2\nu-1)}\right) \mathrm{e}^{2\mathrm{i}\frac{x_0^2}{t}\nu} + \frac{\mathrm{i}t}{2x_0^2}\, \mathrm{e}^{2\mathrm{i}x_0 v_0}\, \mathrm{e}^{2\mathrm{i}\frac{x_0^2}{t}}. \tag{2.19}$$

From (2.14), (2.18) and (2.19) we have

$$(K_2 q)(t) + (K_3 q)(t) = -\mathrm{i}\frac{c_3}{2|x_0|}\, \mathrm{e}^{-4|\alpha||x_0|} t^{3/2}$$
$$\times \int_0^1 \mathrm{d}\sigma \left(q(t(1-\sigma))\, \mathrm{e}^{\mathrm{i}v_0^2 t \frac{(2-\sigma)^2}{2\sigma}} - q_0\right) \sigma^{1/2}\, \mathrm{e}^{\mathrm{i}2|x_0|v_0\frac{(2-\sigma)}{\sigma}}\, \mathrm{e}^{\mathrm{i}\frac{2x_0^2}{t\sigma}} + h_{23}(t) \tag{2.20}$$

where $h_{23}(t) = O(t^2)$.

It is now easy to compute the required coefficients of the expansion.

From (2.6), (2.7), (2.13) and (2.20) we have

$$q_1 = \lim_{t\to 0} \frac{1}{t^{1/2}}(q(t) - q_0) = \lim_{t\to 0}\frac{1}{t^{1/2}}(K_1 q)(t) = -\frac{2\mathrm{i}\beta}{\sqrt{2\pi \mathrm{i}}} q_0. \tag{2.21}$$

Formula (2.21) in particular implies that $(K_2 q)(t) + (K_3 q)(t)$ is $O(t^2)$ (see (2.20)) and then it does not give any contribution to the coefficients we are interested in.

This means that for the computation of q_2 and q_3 we can limit ourselves to analysing (2.7) and (2.13). For q_2 we have

$$q_2 = \lim_{t\to 0}\frac{1}{t}(q(t) - q_0 - q_1 t^{1/2}) = \lim_{t\to 0}\frac{1}{t}(f(t) - q_0) + \lim_{t\to 0}\frac{1}{t}((K_1 q)(t) - q_1 t^{1/2})$$
$$= (|\alpha|v_0 - \mathrm{i}E)q_0 + c_1 \lim_{t\to 0}\frac{1}{t^{1/2}}\int_0^1 \mathrm{d}\sigma \frac{q(t(1-\sigma)) - q_0}{\sqrt{\sigma}}$$
$$= (|\alpha|v_0 - \mathrm{i}E)q_0 + c_1 q_1 \int_0^1 \mathrm{d}\sigma \frac{\sqrt{1-\sigma}}{\sqrt{\sigma}}$$
$$= \left[|\alpha|v_0 + \mathrm{i}\left(\frac{\beta^2}{2} - E\right)\right] q_0. \tag{2.22}$$

Finally, we compute q_3:

$$q_3 = \lim_{t\to 0}\frac{1}{t^{3/2}}(q(t) - q_0 - q_1 t^{1/2} - q_2 t)$$
$$= \lim_{t\to 0}\frac{1}{t^{3/2}}[f(t) - q_0 - (|\alpha|v_0 - \mathrm{i}E)q_0 t] + \lim_{t\to 0}\frac{1}{t^{3/2}}\left[(K_1 q)(t) - q_1 t^{1/2} - \mathrm{i}\frac{\beta^2}{2}q_0 t\right]$$
$$= c_1 q_0 \lim_{t\to 0}\frac{1}{t}\int_0^1 \mathrm{d}\sigma \frac{\mathrm{e}^{\mathrm{i}\frac{v_0^2}{2}t\sigma} - 1}{\sqrt{\sigma}}$$
$$= \frac{\beta v_0^2}{3\sqrt{2\pi \mathrm{i}}} q_0. \tag{2.23}$$

Taking into account that the last term in (2.13) is $O(t^2)$, we conclude the proof of the proposition. □

3. Ionization probability for small times

In this section, we shall use the result of proposition 2.1 to give the asymptotic expansion for the ionization probability $\mathcal{P}(t)$.

Using expansion (2.2) in (1.16) we have

$$\begin{aligned} A(t) &= \mathrm{i}\beta q_0 \int_0^t \mathrm{d}s (q_0 + q_1 s^{1/2} + q_2 s + q_3 s^{3/2} + O(s^2))(1 + (|\alpha| v_0 + \mathrm{i}E)s + O(s^2)) \\ &= \mathrm{i}\beta q_0^2 t + \frac{2}{3}\mathrm{i}\beta q_0 q_1 t^{3/2} + \frac{1}{2}\beta q_0(-Eq_0 + \mathrm{i}|\alpha| v_0 q_0 + \mathrm{i}q_2)t^2 \\ &\quad + \frac{2}{5}\beta q_0(-Eq_1 + \mathrm{i}|\alpha| v_0 q_1 + \mathrm{i}q_3)t^{5/2} + O(t^3). \end{aligned} \tag{3.1}$$

From (3.1) we obtain

$$\begin{aligned} \mathrm{Re}\, A(t) = &-\tfrac{2}{3}\beta q_0 \,\mathrm{Im}\, q_1 t^{3/2} - \tfrac{1}{2}\beta q_0 (Eq_0 + \mathrm{Im}\, q_2)t^2 \\ &- \tfrac{2}{5}\beta q_0 (E\,\mathrm{Re}\, q_1 + |\alpha| v_0 \,\mathrm{Im}\, q_1 + \mathrm{Im}\, q_3)t^{5/2} + O(t^3) \end{aligned} \tag{3.2}$$

and

$$|A(t)|^2 = \beta^2 q_0^4 t^2 + \tfrac{4}{3}\beta^2 q_0^3 \,\mathrm{Re}\, q_1 t^{5/2} + O(t^3). \tag{3.3}$$

Inserting (3.2), (3.3) in (1.17) and using the explicit expressions for $q_k, k = 0, 1, 2, 3$ (see (1.13), (2.3), (2.4) and (2.5)) we find

$$\begin{aligned} \mathcal{P}(t) &= -\frac{4}{3}\beta q_0 \,\mathrm{Im}\, q_1 t^{3/2} - \beta q_0 \big(Eq_0 + \mathrm{Im}\, q_2 + \beta q_0^3\big)t^2 \\ &\quad - \frac{4}{5}\beta q_0 \left(E\,\mathrm{Re}\, q_1 + |\alpha| v_0 \,\mathrm{Im}\, q_1 + \mathrm{Im}\, q_3 + \frac{5}{3}\beta q_0^2 \,\mathrm{Re}\, q_1\right) t^{5/2} + O(t^3) \\ &= \beta^2 |\alpha| \,\mathrm{e}^{-2|\alpha||x_0|} \left[\frac{4}{3\sqrt{\pi}} t^{3/2} - \frac{1}{2}(\beta + 2|\alpha|\,\mathrm{e}^{-2|\alpha||x_0|})t^2 \right. \\ &\quad \left. + \frac{4}{5\sqrt{\pi}} \left(|\alpha| v_0 - \frac{\alpha^2}{2} + \frac{v_0^2}{6} + \frac{5}{3}\beta|\alpha|\,\mathrm{e}^{-2|\alpha||x_0|}\right) t^{5/2}\right] + O(t^3). \end{aligned} \tag{3.4}$$

We note that the first two terms of the expansion depend on various parameters (strength of the interaction, energy of the bound state, initial position of the time-dependent potential) but they do not depend on the initial velocity v_0.

The first non-trivial term depending on v_0 is contained in the coefficient of $t^{5/2}$. As it should be expected, the value of such coefficient computed for $v_0 > 0$ is larger than that computed for $v_0 < 0$.

Remark. We do not know at this stage whether the dependence on v_0 of the ionization probability for small times appears in the term of order $t^{5/2}$ also in the genuine quantum two-body problem. What we expect is that such dependence will not appear in the first non-trivial term of the expansion.

We conclude this paper with few comments about the relevance that, in our opinion, solvable models of the kind described above have in the description of the behaviour of quantum systems in interaction with a large environment.

In a seminal paper dating back to the early days of quantum mechanics [M], Mott tried to identify the relevant aspects of the motion of a quantum particle in a cloud chamber. In particular, he was interested in the reasons why a quantum particle emitted by a source as a spherical wave packet reveals itself with a straight track in the cloud chamber surrounding the source. In his words, 'It is a little difficult to picture how it is that an outgoing spherical wave can produce a straight track; we think intuitively that it should ionize at random throughout space'.

The idea of Mott was to consider the simplest situation in which, roughly speaking, the definition of a probability conditioned to the occurrence of a specific state of the environment was possible.

More precisely, he studied a quantum particle in the presence of two atoms whose electrons were initially in their ground states. The initial state of the quantum particle was assumed to be a spherically symmetric outgoing wave.

Performing a perturbative analysis of the time-independent Schrödinger equation he computed the ionization probability of the second atom, assuming that the first atom was already ionized. He found that this probability was much higher if the position of the second atom were on the line connecting the centre of the initial wave packet of the quantum particle with the position of the first atom.

A rigorous approach to the problem outlined above is by no means trivial since it involves a detailed analysis of a quantum three-body problem.

A significant simplification is obtained assuming that the quantum particle is much heavier than the electrons. In such a case, the evolution of the quantum particle would be almost free and the dynamics of the two electrons would be essentially decoupled.

Note that in a one-dimensional realization of the model the spherical wave would be replaced by a coherent superposition of two wave packets starting from the same point x_0 with opposite (average) velocities $\pm v_0$.

In this rough approximation, one is limited to studying a non-autonomous evolution problem of the kind analysed here.

It should be emphasized that in our case the problem is further simplified by the use of point interactions. In fact, the availability of the unitary propagator U_α in closed form makes the computations much easier and in turn the entire analysis becomes more transparent.

In the light of these final comments, we consider our result as a very preliminary step towards a rigorous treatment of a model describing the motion of a quantum particle in a one-dimensional cloud chamber.

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